ON THE THREE-DIMENSIONAL SONIC FLOW OF AN IDEAL GAS PAST A BODY

PMM Vol. 32, No. 2, 1968, pp. 285-290

V.N. DIESPEROV and O.S. RYZHOV
(Moscow)

(*Received December 29, 1967*)

Guderley, Joshihara and Barish [1 and 2] were the first workers to present a solution to the problem of decay of perturbations at some distance from a finite body in a sonic flow of an ideal gas (i.e. nonviscous and without heat conductivity). They had to resort to numerical methods of integrating ordinary differential equations in order to determine completely the flow parameters. Using the parametric representation, Fal'kovich and Chernov [3] succeeded in obtaining the unknown functions in the closed appearance as well, as the self-similar variable in its exact form. Analogous results were also obtained by Müller and Matschat in [4].

In all the above works it was assumed that both, the body and the velocity field of the perturbed flow were axially symmetric. In the case of flows which are substantially three-dimensional we find, that, although the principal term of the solution governing the asymptotic laws of decay of perturbations remains unchanged, it requires additional correction terms accounting for the changes in the parameters of the medium in the direction of the angular coordinate [5]. Euvrard in [6] gave the form of these corrections for the region in front of the shock wave. Below we construct a solution relevant to the flow behind the shock wave and establish the connection between this flow and the lift, which together with the side force act on the body.

1. We shall assume that no dissipative processes caused by viscosity and heat conductivity take place in the flow and, that within the approximation used, the flow is isentropic. Let the flow originating at infinity move with the critical velocity a_* along the x-axis of the cylindrical (x, r, Θ)-coordinate system. Since the velocity field is vortex-free, we can pass directly from the Euler's system of equations to a single partial differential equation for the potential ϕ. We know that

$$\left[a^2-\left(\frac{\partial\varphi}{\partial x}\right)^2\right]\frac{\partial^2\varphi}{\partial x^2}+\left[a^2-\left(\frac{\partial\varphi}{\partial r}\right)^2\right]\frac{\partial^2\varphi}{\partial r^2}+\frac{1}{r^2}\left[a^2-\frac{1}{r^2}\left(\frac{\partial\varphi}{\partial\Theta}\right)^2\right]\frac{\partial^2\varphi}{\partial\Theta^2}- \tag{1.1}$$

$$-2\frac{\partial\varphi}{\partial x}\frac{\partial\varphi}{\partial r}\frac{\partial^2\varphi}{\partial x\,\partial r}-\frac{2}{r^2}\frac{\partial\varphi}{\partial x}\frac{\partial\varphi}{\partial\Theta}\frac{\partial^2\varphi}{\partial x\,\partial\Theta}-\frac{1}{r^2}\frac{\partial\varphi}{\partial\Theta}\frac{\partial\varphi}{\partial r}\left(2\frac{\partial^2\varphi}{\partial r\,\partial\Theta}-\frac{1}{r}\frac{\partial\varphi}{\partial\Theta}\right)+\frac{a^2}{r}\frac{\partial\varphi}{\partial r}=0$$

$$\frac{1}{2}(v_x^2+v_r^2+v_\Theta^2)+w=\frac{1}{2}a_*^2+w_*\qquad\left(v_x=\frac{\partial\varphi}{\partial x},\ v_r=\frac{\partial\varphi}{\partial r},\ v_\Theta=\frac{1}{r}\frac{\partial\varphi}{\partial\Theta}\right) \tag{1.2}$$

where v_x, v_r and v_Θ are the relevant velocity components, a denotes the velocity of sound, w is the specific enthalphy and an asterisk denotes the critical values of the gas parameters. An expression for the velocity of sound in terms of partial derivatives of the potential with respect to the coordinates, is obtained from the Bernoulli integral (1.2) using the equation of state expressing the pressure p as a function of the specific volume V (or the density $\rho = 1/V$), and the specific entropy s. We have, with the required accuracy

$$a = a_* + \left(\frac{\partial a}{\partial w_*}\right)_s (w - w_*) + \cdots$$

In an adiabatic process the specific enthalphy increment is $dw = Vdp$, hence

$$\left(\frac{\partial a}{\partial w}\right)_s = \left(\frac{\partial a}{\partial \rho}\right)_s \left(\frac{\partial \rho}{\partial p}\right)_s \left(\frac{\partial p}{\partial w}\right)_s = \frac{m-1}{a} \qquad \left(m = \frac{1}{2\rho^3 a^2}\left(\frac{\partial^2 p}{\partial V^2}\right)_s\right)$$

We note that although the stream is bisected by a shock wave [2], the latter is a weak one and the associated entropy change has a higher order of smallness in comparison to the magnitudes which are taken into account in the approximation considered. In the following we shall denote by the subscript 1 the gas parameters in front of the shock wave, and by 2 those behind the shock wave. By v_n and v_τ we shall denote the velocity components, normal and tangential to the shock front. We find that, within the used approximation, two of the Hugoniot [7] conditions are satisfied automatically on the passage across the shock. According to the first of them, pressure and density are connected by an adiabatic relation with the accuracy of up to the second order of smallness

$$p_2 - p_1 = \left(\frac{\partial p}{\partial \rho_1}\right)_s (\rho_2 - \rho_1) + \frac{1}{2}\left(\frac{\partial^2 p}{\partial \rho_1^2}\right)_s (\rho_2 - \rho_1)^2 + \cdots \tag{1.3}$$

The other condition is, that the pressure behind the shock can be expressed in terms of the pressure in front of the shock, using the Bernoulli integral which remains also valid for the flows exhibiting discontinuities. Moreover, the product of both normal velocity components is governed by the following equality [7]:

$$v_{n1} v_{n2} = \frac{p_2 - p_1}{\rho_2 - \rho_1}$$

Using the expansion (1.3) and introducing the specific enthalphy w as an independent thermodynamic variable, we obtain

$$v_{n2} v_{n1} = a_*^2 + (m_* - 1)(w_2 + w_1 - 2w_*) + \cdots \tag{1.4}$$

The last of the Hugoniot conditions requires the continuity of the tangential velocity component v_τ on passing through the shock front. It can be replaced with the condition of continuity of the potential

$$\varphi_2 = \varphi_1 \tag{1.5}$$

2. We shall seek a solution of (1.1) in the form of the following expansion:

$$\varphi = a_*\left[x + \sum_i \varphi_{\omega_i}(x, r, \Theta)\right], \quad \varphi_{\omega_i}(x, r, \Theta) = r^{\omega_i} f_{\omega_i}(\xi, \Theta), \quad \xi = \frac{x}{(2m_*)^{1/3} r^n} \tag{2.1}$$

and we shall define the position of the shock wave thus:

$$\xi = \xi_s \left[1 + \sum_i \xi_{\sigma_i}(r, \Theta)\right], \qquad \xi_{\sigma_i}(r, \Theta) = r^{\sigma_i} c_{\sigma_i}(\Theta) \tag{2.2}$$

Function ϕ_{ω_0} is the integral of the approximate von Kármán equation [8] and it yields the laws of decay of perturbation at some distance from an arbitrary finite body in a uniform sonic flow. The shock wave tends asymptotically to the surface of revolution $\xi = \xi_s$ = const It was shown rigorously by Fal'kovich and Chernov in [3] that, in the expansion (2.1), $n = 4/7$ and the first power index $\omega_0 = -2/7$. The function $f_{-2/7} = f_{-2/7}(\xi)$ can be defined in the mixed sub- and supersonic region in front of the shock wave, with help of the following parametric formulas

$$\xi = \frac{12\eta - 5}{7\eta^{3/7}}, \quad f_{-2/7} = 2^5 \cdot 7^{-3} \cdot \eta^{1/7}(12\eta^2 - 15\eta - 25) \tag{2.3}$$

where the normalization is performed so, that the limiting characteristic flow surface corresponds, in the first approximation, to $\xi = \eta = 1$. Behind the shock wave we have

$$\xi = b^{3/7} \frac{12\zeta + 5}{7\zeta^{3/7}}, \quad f_{-2/7} = 2^5 \cdot 7^{-3} \cdot b^{1/7}\zeta^{1/7}(12\zeta^2 + 15\zeta - 25) \tag{2.4}$$

According to Euvrard [9] we have

$$b = (2 - \sqrt{3})^{1/3}, \quad \xi_s = 2\cdot 3^{1/7} (\sqrt{3} - 1)^{1/7}$$

and the values of the parametric variables are

$$\eta_s = 12^{-1} (7\sqrt{3} + 12), \quad \zeta_s = 12^{-1} (7\sqrt{3} - 12)$$

Each subsequent function f_{ω_ι} is obtained by solving a linear differential equation which may be homogeneous or nonhomogeneous, depending on the value of ι. It can be shown that $\omega_1 = -4/7$, while the function $f_{-4/7} = f_{-4/7}(\xi)$ is connected with the drag acting on the body. The equation governing this function is homogeneous

$$\left(\frac{df_{-2/7}}{d\xi} - \frac{16}{49}\xi^2\right)\frac{d^2 f_{-4/7}}{d\xi^2} + \left(\frac{d^2 f_{-2/7}}{d\xi^2} - \frac{48}{49}\xi\right)\frac{df_{-4/7}}{d\xi} - \frac{16}{49} f_{-4/7} = 0 \tag{2.5}$$

From [5 and 9] it follows that $f_{-4/7}$ should be identically equal to zero in the region in front of the shock wave, otherwise the flow velocity field would have a singularity on the limit characteristic surface. To obtain the form of $f_{-4/7}$ in the region behind the shock wave we must use (1.4) and (1.5) and put, in (2.2), $\sigma_0 = \omega_0 = -2/7$ and $c_{-2/7} =$ const. The first condition

$$f_{-4/7,\,2} = c_{-2/7}\xi_s \left(\frac{df_{-2/7,\,1}}{d\xi} - \frac{df_{-2/7,\,2}}{d\xi}\right) \tag{2.6}$$

requires the continuity of potential across the shock wave, while the second condition follows from the relationship (1.4). Combining this condition with (2.6) yields

$$\left(\frac{df_{-2/7,\,2}}{d\xi} - \frac{16}{49}\xi_s^2\right)\frac{df_{-4/7,\,2}}{d\xi} - \frac{16}{49}\xi_s f_{-4/7,\,2} = 0 \tag{2.7}$$

We note that the left-hand side of (2.5) is the derivative of the left-hand side of (2.7), with the subscript 2 of the functions $f_{-2/7}$ and $f_{-4/7}$ and the subscript s of ξ omitted. This yields the required integral

$$f_{-4/7} = 2^{-44/35}\cdot 3^{-12/35}\cdot 7^{2/5}(7 - 4\sqrt{3})^{2/7} A \exp\left(16\int_{\xi_s}^{\xi}\frac{\xi\, d\xi}{49\, df_{-2/7}/d\xi - 16\xi^2}\right)$$

Here A denotes an arbitrary constant. We can obtain the above integral more simply by replacing ξ with ζ according to the first formula of (2.4). Indeed,

$$f_{-4/7} = A\zeta^{3/7}(\zeta + 1)^{2/3}$$

Condition (2.6) serves to determine the value of the constant $c_{-2/7}$. Subsequent magnitude ω_2 in (2.1) is equal to $-6/7$ and $f_{-6/7} = f_{-6/7}(\xi)$ as before. In Eq. (2.2) the parameter $\sigma_1 = \omega_1 = -4/7$, while $c_{-4/7} =$ const. The resulting ordinary differential equation for $f_{-6/7}$ is nonhomogeneous, since the product $(df_{-4/7}/d\xi)(d^2 f_{-4/7}/d\xi^2)$ enters its right-hand side This is easily integrable, because the required solution $f^0_{-6/7}$ of the corresponding homogeneous equation is [5]

$$f^0_{-6/7} = B\frac{df_{-2/7}}{d\xi} \qquad (B = \text{const})$$

3. We shall now obtain the first terms of the expansions (2.1) and (2.2) which, as a matter of fact, to a three-dimensional flow past a finite body. We have shown before that the functions $\phi_{-2/7}$, $\phi_{-4/7}$ and $\phi_{-6/7}$ allow us to construct only those gas flows, which possess axial symmetry. To simplify the required solution even more, we shall expand it into a Fourier series, retaining only the terms containing first harmonics. We have

$$f_{\omega_3} = \psi_{\omega_3}(\xi)(c_y\cos\Theta + c_z\sin\Theta), \quad c_{\sigma_2} = d_{\sigma_2}(c_y\cos\Theta + c_z\sin\Theta)_2, \quad \sigma_2 = \omega_3 + 2/7 \tag{3.1}$$

The parameter ω_3 can be found from the condition that the velocity components have no singularities on the limit characteristic surface. This implies that the function ψ_{ω_3} must be regular when $\xi \to 1$. Euvrard has shown in [6] that the latter requirement implies that $\omega_3 = -1$ and hence $\sigma_2 = \omega_2 + 1/7 = -5/7$. This result can be obtained by another method utilizing the relation between the spatial type perturbations and the lift, acting together with the side force on the body.

Let us denote the lift by F_y and the side force by F_z and represent them by

$$F_y = F_y' + F_y'', \qquad F_z = F_z' + F_z'' \tag{3.2}$$

Magnitudes F_y' and F_z' are connected with the loss of impulse by the gas in the vortex wake which is always formed behind the body [7]. The presence of the terms F_y'' and F_z'' in the right-hand sides of (3.2) is caused by the fact that the y- and z-components of the momentum can be transported to infinity not only by the vortex wake but also by means of the system of waves spreading from the body into the external stream. In order to find F_y'' and F_z'' we shall construct a control cylindrical surface around the body, its radius equal to r and with its generators parallel to the x-axis. This will enable us to utilize the integral of the tensor of the impulse flux density taken over this surface [7] in calculating the forces. It can be shown that major contribution to this integral is made by the terms proportional to the pressure, while the contribution of the terms incorporating the velocities of the gas particles will be smaller by an order of magnitude. Thus

$$F_y'' = -\int_{-\infty}^{+\infty}\int_0^{2\pi} (p - p_*)\, r \cos\theta \, d\theta \, dx, \qquad F_z'' = -\int_{-\infty}^{+\infty}\int_0^{2\pi} (p - p_*)\, r \sin\theta \, d\theta \, dx$$

Expanding the pressure in terms of the specific enthalphy and applying the Bernoulli integral to the expression obtained we find, that

$$F_y'' = \pi\rho_* a_*^2 c_y r^{\omega_3+1} D, \qquad F_z'' = \pi\rho_* a_*^2 c_z r^{\omega_3+1} D$$
$$D = \int_{-\infty}^{+\infty} \frac{d\psi_{\omega_3}}{d\xi} d\xi + \xi_s d_{\sigma_2} \left(\frac{df_{-2/7,1}}{d\xi} - \frac{df_{-2/7,2}}{d\xi}\right) \tag{3.3}$$

Let now $r \to \infty$. To retain F_y'' and F_z'' finite and different from zero we must put $\omega_3 = -1$, since at this value of ω_3, as we said before, the solution has no singularities on the limit characteristic surface [6].

Let us derive an explicit expression for ψ_{-1} satisfying

$$\left(\frac{df_{-2/7}}{d\xi} - \frac{16}{49}\xi^2\right)\frac{d^2\psi_{-1}}{d\xi^2} + \left(\frac{d^2 f_{-2/7}}{d\xi^2} - \frac{72}{49}\xi\right)\frac{d\psi_{-1}}{d\xi} = 0 \tag{3.4}$$

Only first and second derivatives of the required function are present in the above expression, therefore a general integral dependent on the constants H_1 and H_2 can be written down at once as

$$\psi_{-1} = H_1 + H_2 \int \exp\left(-\int \frac{49 d^2 f_{-2/7}/d\xi^2 - 72\xi}{49\, df_{-2/7}/d\xi - 16\xi^2} d\xi\right) d\xi$$

This expression can be considerably simplified by passing to the parametric variables. By (2.3) we have, in front of the shock wave,

$$\frac{d^2 f_{-2/7}}{d\xi^2} - \frac{72}{49}\xi = -\frac{2^3 \cdot 3 \cdot 5\,(4\eta^2 - \eta - 3)}{7^3 \eta^{5/7}(6\eta + 1)}, \qquad \frac{df_{-2/7}}{d\xi} - \frac{16}{49}\xi^2 = \frac{2^4 \cdot 5^2 (6\eta + 1)(\eta - 1)}{7^4 \eta^{4/7}}$$

Denoting the arbitrary constants by H_{11} and H_{21}, we have

$$\psi_{-1} = H_{11} + H_{21}\eta \tag{3.5}$$

For the region behind the shock wave we obtain, in the analogous manner,

$$\frac{d^2 f_{-2/7}}{d\xi^2} - \frac{72}{49}\xi = \frac{2^3 \cdot 3 \cdot 5 \cdot b^{3/7}(4\zeta^2 + \zeta - 3)}{7^3 \zeta^{5/7}(6\zeta - 1)}, \quad \frac{df_{-2/7}}{d\xi} - \frac{16}{49}\xi^2 = \frac{2^4 \cdot 5^2 \cdot b^{6/7}(6\zeta - 1)(\zeta + 1)}{7^4 \zeta^{4/7}}$$

and

$$\psi_{-1} = H_{12} + H_{22}\zeta \tag{3.6}$$

with different values for the constants H_{12} and H_{22}. We shall first show that the value $H_{11} = 0$ must be chosen in (3.5). Indeed, otherwise both, radial and angular velocity components tend, in the region in front of the shock wave, to infinity as r^{-2} when the x-axis is approached at any point. If $H_{11} = 0$, we can easily see that $v_r \sim v_\Theta \sim H_{21} |x|^{-7/2}$. The longitudinal velocity component $v_x \sim H_{21} |x|^{-9/2} r$, also exhibits no singularities. In the region behind the shock wave v_r and v_Θ exhibit a singularity noted previously, when $H_{12} \neq 0$. In this region however, we cannot impose the requirement of the absence of singularities in the flow velocity field near the axis $r = 0$, since we have a vortex wake behind the body, where the solution (3.6) is invalid. On the outer boundary of the vorticity region the solution (3.6) should be correlated with a solution allowing for the dissipative processes taking place inside the wake. Landau and Lifshits [7] used this idea in connection with the incompressible fluid motions. In [10] an analogous method was used to investigate a flow of a viscous, heat conducting gas moving with sonic velocity at infinity, and an integral was obtained describing the velocity field in the wake at large distances from the body. Without going deeper into the matter we shall indicate that the results of [10] readily yield the relationship between the products of H_{12} and the constants c_y and c_z from the right-hand sides of (3.1), and the magnitudes governing the laws of degeneracy of perturbations within the wake.

Let us now turn to the boundary conditions which should hold on passing across the front of the shock wave. Inserting (2.1) and (2.2) into (1.5) and taking into account the form of f_{-1} given by (3.1) we have, in terms of the parametric variables,

$$H_{21}\eta_s = H_{12} + H_{22}\zeta_s - \xi_s d_{-3/7} \left(\frac{df_{-2/7,1}}{d\xi} - \frac{df_{-2/7,2}}{d\xi} \right) \tag{3.7}$$

The second condition follows from (1.4). To calculate v_{n2} and v_{n1} we must know the expression for the projection of the unit vector $\mathbf{n}$ normal to the shock front, on the x-axis. The other components of $\mathbf{n}$ are not required in this approximation. Elementary manipulations yield

$$n_x = 1 - {}^8/_{49} (2m_*)^{2/7} \xi_s^2 \left[r^{-6/7} + c_{-2/7} r^{-8/7} + {}^1/_4 c^2_{-2/7} r^{-10/7} - {}^1/_2 d_{-3/7} (c_y \cos\Theta + c_z \sin\Theta) r^{-11/7} \right]$$

from which we find

$$\frac{d\psi_{-1,2}}{d\xi} + \frac{d\psi_{-1,1}}{d\xi} = - \xi_s d_{-3/7} \left(\frac{16}{49} \xi_s + \frac{d^2 f_{-2/7,1}}{d\xi^2} + \frac{d^2 f_{-2/7,2}}{d\xi^2} \right) \tag{3.8}$$

Condition (3.7) enables us to determine the constant D appearing in (3.3), for the lift and the side force. We have

$$D = H_{21}\eta_s - H_{22}\zeta_s + \xi_s d_{-3/7} \left(\frac{df_{-2/7,1}}{d\xi} - \frac{df_{-2/7,2}}{d\xi} \right) = H_{12}$$

From this we conclude that the removal of the y- and z-components of the momentum to infinity resulting in the fact that the contribution of the perturbations of the outer flow towards the establishment of transverse forces is not nil, is connected with obligatory appearance of the singularities in the v_r and v_Θ velocity components at the points lying on the positive part of the x-axis. These singularities disappear when $H_{12} = 0$, but then F_y'' and F_z'' also become equal to zero. Using the results of [10] we can show, that the total force acting on the body in the direction perpendicular to the incident flow, is also equal to zero. When $H_{12} = 0$, the wake is axisymmetric in the first approximation, in spite of the fact that, as we have shown, the perturbations in the flow may be three-dimensional. To conclude the solution of the problem, we obtain from (3.8), the constant

$$d_{-3/7} = - 2^{-47/7} \cdot 3^{-1/7} \cdot 5^{-2} \cdot 7^3 (2 - \sqrt{3})^{-3/7} \left[H_{21}\eta_s (2 - \sqrt{3}) - H_{22}\zeta_s \right]$$

BIBLIOGRAPHY

1. Guderley, K.G. and Joshihara, H., An axial-symmetric transonic flow pattern. Quart. Appl. Math., Vol. 8, No. 4, 1951.
2. Barish, D.T. and Guderley, K.G., Asymptotic forms of shock waves in flows over symmetrical bodies at Mach 1. J. Aeronaut. Sci., Vol. 20, No. 7, 1953.
3. Fal'kovich, S.V. and Chernov, I.A., Flow of a sonic gas stream past a body of revolution. PMM Vol. 28, No. 2, 1964.
4. Müller, E.A. and Matschat, K., Ähndlichkeitslösungen der transsonischen Gleichungen bei der Anström – Machzahl 1. Proc. 11-th Interbat. congr. of appl. mech., Munich, 1964, Springer – Ver, 1966.
5. Guderley, K.G., Theory of Transonic Flows. M., Izd. inostr. lit., 1960.
6. Euvrard, D., Ecoulement asymptotique à grande distance d'un obstacle transsonique tridimensionnel. C. r. Acad. sci., Sér. A, t. 262, No. 4, 1966.
7. Landau, L.D. and Lifshits, E.M., Mechanics of Continuous Media. M., GITTL, 1954.
8. Ka'rmán, Th. The similarity law of transonic flow. J. Math and Phys., Vol. 26, No. 3, 1947.
9. Euvrard, D. Ecoulement transsonique à grande distance d'un corps de révolution. C. r. Acad. sci., Groupe 2, t. 260, No. 22, 1965.
10. Ryzhov, O.S. and Terent'ev, E.D., On perturbations associated with the creation of lift acting on a body in a transonic stream of a dissipative gas. PMM Vol. 31, No. 6, 1967.

Translated by L.K.